

# Quantum Processes as Thermodynamic Resources: The Role of Non-Markovianity

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Quantum thermodynamics studies how quantum systems and operations may be exploited as sources of work to perform useful thermodynamic tasks. In real-world conditions, the evolution of open quantum systems typically displays memory effects, resulting in a non-Markovian dynamics. The associated information backflow has been observed to provide advantage in certain thermodynamic tasks. However, a general operational connection between non-Markovianity and thermodynamics in the quantum regime has remained elusive. Here, we analyze the role of non-Markovianity in the central task of extracting work via thermal operations from general multitime quantum processes, as described by process tensors. By defining a hierarchy of four classes of extraction protocols, expressed as quantum combs, we reveal three different physical mechanisms (work investment, multitime correlations, and system-environment correlations) through which non-Markovianity increases the work distillable from the process. The advantages arising from these mechanisms are linked precisely to a quantifier of the non-Markovianity of the process. These results show in very general terms how non-Markovianity of any given quantum process is a fundamental resource that unlocks an enhanced performance in thermodynamics.

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*Introduction*—Designing efficient strategies to extract work has been the cornerstone of thermodynamics since the dawn of the first industrial revolution. The advent of *quantum thermodynamics* [1] has opened new scenarios to identify fundamental mechanisms, spanning from coherence and correlations to reservoir engineering and even ignorance [2–6], which may be exploited to extract additional work from quantum systems. The modern formulation of quantum thermodynamics as a resource theory [7–10] in fact provides an operational justification for the focus on work extraction: given that the theory is asymptotically reversible both at static [8,11,12] and dynamic [13–15] levels, any state or channel transformation is fully characterized in the asymptotic limit by its work content. We are thus faced with the key question: how can we maximize work extraction from general quantum processes?

When investigating quantum properties of general thermodynamic tasks, Markovianity of the underlying dynamics is often assumed for ease of analytical calculations, but such a simplification comes with a price to pay.

That is, in a Markovian dynamics any resource the system loses to the environment at some point in time cannot be later recovered and consumed for the task. Conversely, non-Markovian dynamics allows for such a resource backflow [16], which has been considered as a potential source of advantage in a number of thermodynamic tasks [17–32]. While for a continuous-time evolution these phenomena can be explored using tools such as non-Markovian master equations [33–35], dynamical maps [36], or continuous thermomajorization [37–40], to conclusively investigate the role of non-Markovianity in thermodynamics we may need to resort to an alternative approach.

Let us adopt a practical standpoint where an experimenter can access the system at a discrete set of times. The dynamics is then better described by means of the *process tensor* framework [41]. A desirable feature of this approach is that the quantum comb structure [42] of process tensors allows a natural definition of quantum (non-)Markovianity [43], removing any ambiguity present in other approaches [44–49]. This makes process tensors useful for understanding memory effects in quantum dynamics [50–64] and revisiting problems that had mostly been treated under the Markov hypothesis, such as quantum process tomography [65–70], simulation [71–77], and thermalization [78–83], among others [84–89].

In this Letter, we establish non-Markovianity of quantum processes as a fundamental resource for work extraction in thermodynamics. Building on recent results for work cost and work distillable from quantum states [8,11,12] and

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channels [13–15], we quantify the net work  $W$  extractable from general multitime quantum process tensors. We define a hierarchy of four classes of strategies [90], labeled, respectively, as *sequential*, *joint*, *global*, and *comb*, which enable progressively more work to be extracted. Crucially, for a Markovian process, these strategies are all equivalent

$$\begin{array}{ccccccc}
 W^{\text{seq}} & \leq & W^{\text{joint}} & \leq & W^{\text{global}} & \leq & W^{\text{comb}} \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \text{work} & & \text{multitime} & & \text{system-environment} & \\
 & \text{investment} & & \text{correlations} & & \text{correlations} & 
 \end{array} \quad (1)$$

Although these mechanisms may coexist in general, we provide examples in which their action is isolated, validating our definitions and interpretation. We also derive quantitative bounds precisely linking the enhancement provided by each mechanism to the amount of non-Markovianity present in the process. Our results bridge the gap between quantum dynamics and thermodynamics, providing a fundamental characterization of the advantages unlocked by non-Markovian processes in thermodynamic tasks.

*Setup*—We consider the resource-theoretic scenario in which an experimenter can perform *thermal operations* for free to implement thermodynamic tasks [7–10]. That is, given a bath  $B$  with Hamiltonian  $H_B$  and temperature  $T$  and a system  $S$  with (time-independent) Hamiltonian  $H_S$  [91], the operations  $\mathcal{E}_{TO}$  the experimenter can perform on the state  $\rho_S$  of the system are of the form

$$\mathcal{E}_{TO}(\rho_S) = \text{tr}_B[U(\rho_S \otimes \gamma_B)U^\dagger], \quad (2)$$

where  $\gamma_B := e^{-H_B/kT}/\text{tr}[e^{-H_B/kT}]$  is the bath thermal state,  $k$  is Boltzmann’s constant, and  $U$  is a unitary operator that acts jointly on system and bath ensuring conservation of the total energy,  $[U, H_S + H_B] = 0$ .

The goal of the experimenter is to extract, by means of thermal operations, the maximum amount of work from any given nonthermal resource—which could be a state, a channel, or a multitime process—in the asymptotic setting in which  $n \rightarrow \infty$  copies of the given resource are available.

and the hierarchy collapses. On the other hand, for a general process we identify the three mechanisms of work investment, multitime correlations, and system-environment correlations, through which non-Markovianity strictly enhances work extraction at every step up the hierarchy:

The work  $W(\rho_S)$  asymptotically distillable from a general state  $\rho_S$  is known to be given by  $kTS(\rho_S||\gamma_S)$ , where  $S(\rho||\sigma) = \text{tr}[\rho(\ln\rho - \ln\sigma)]$  is the relative entropy between the states  $\rho$  and  $\sigma$  [8,11,12]. Reference [8] also showed that state conversion is asymptotically reversible, implying that the work needed to transform a thermal state into another state  $\rho$  is also given by  $W(\rho)$ . Similarly, work extraction from a nonthermal channel  $\mathcal{E}$  was discussed in Ref. [13], which showed that the best protocol consists of preparing some optimal state  $\rho$  at a cost  $W(\rho)$ , then using the channel to obtain the state  $\mathcal{E}(\rho)$ , and finally extracting  $W(\mathcal{E}(\rho))$  of work in the asymptotic limit. In this way, the maximum work distillable from a channel  $\mathcal{E}$  is given by  $W(\mathcal{E}) := \max_\rho \{W[\mathcal{E}(\rho)] - W(\rho)\}$ .

We now take the next step in this direction and analyze work extraction from general multitime quantum processes. Consider that the experimenter can prepare an initial state and then perform a sequence of control operations on the system. In between these operations the system interacts with an uncontrolled environment, giving rise to a multitime open quantum system dynamics. Such a dynamics is described by a process tensor  $\mathcal{P}$  mapping the sequence of control operations  $\mathcal{S}$  to the final state of the system [41]. The physical constraints on the process imply that, for a process with an initial state preparation and  $n - 1$  control operations,  $\mathcal{P}$  belongs to the set  $\mathbb{P}_n$  of  $n$ -step *quantum combs* [42]. Consequently, we have  $\mathcal{S} \in \mathbb{S}_n$ , where  $\mathbb{S}_n$  is the set of quantum combs mapped by the elements of  $\mathbb{P}_n$  to a final state  $\sigma = \mathcal{P}(\mathcal{S})$ , as in Fig. 1(a).

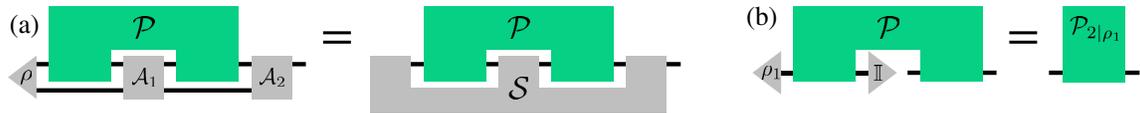


FIG. 1. (a) General multitime quantum process. The experimenter prepares an initial state  $\rho$  for the system and then performs the control operations described by the channels  $\mathcal{A}_1$  and  $\mathcal{A}_2$ . In between operations the system interacts with an uncontrolled environment. The dynamics is described by a process tensor  $\mathcal{P}$  mapping the control operations to the final state of the system. The physical constraints on the process imply  $\mathcal{P}$  is a quantum comb. Since the control operations may also be correlated by an ancilla, they are also in general described by a comb  $\mathcal{S}$ . (b) Channel  $\mathcal{P}_{2|\rho_1}$  associated with the second step of the dynamics described by  $\mathcal{P}$  once the input state  $\rho_1$  of the first step is specified and the output discarded. In general, the channel describing the evolution in any given step of the dynamics will be conditioned on all previous inputs, but never on the subsequent ones since the process is time ordered.

Importantly, the structure of  $\mathcal{P}$  is also the same as that of a sequence of quantum channels with memory [92], so it can be seen as an ordered mapping from a set of input states to a set of output states. In this sense, while the first step of the process is always a proper channel  $\mathcal{P}_1$ , the subsequent steps may be channels conditioned on the inputs at previous times, as in Fig. 1(b). However, if that is not the case and the process consists of a sequence of independent channels, we say the process is *Markovian*, as no memory is carried along the dynamics [43]. This implies that extracting work from a Markovian process is the same as extracting work from a set of channels, for which one can apply the protocol of Ref. [13] and the work distillable from the process will simply be the sum of the works distillable from each channel. Therefore, any fundamental difference between extracting work from channels and general processes can only come from non-Markovian effects. In this sense, it is crucial to relate the amount of extractable work to the amount of non-Markovianity of the process.

To quantify the non-Markovianity of any given process  $\mathcal{P} \in \mathbb{P}_n$ , we employ the following distinguishability measure with respect to the closest Markovian process:

$$N(\mathcal{P}) := \min_{\mathcal{Q} \in \mathbb{P}_n^M} \bar{S}(\mathcal{P}||\mathcal{Q}), \quad (3)$$

where  $\mathbb{P}_n^M$  is the set of Markovian  $n$ -step process tensors and

$$\bar{S}(\mathcal{P}||\mathcal{Q}) := \max_{\mathcal{S} \in \mathcal{S}_n} S[\mathcal{P}(\mathcal{S})||\mathcal{Q}(\mathcal{S})] \quad (4)$$

is the relative entropy between process tensors  $\mathcal{P}$  and  $\mathcal{Q}$  [89]. This is a fully bona fide measure fulfilling the prescriptions demanded by the resource theory of non-Markovianity [61].

Having set the stage with background results and definitions, we now proceed to the hierarchy (1) of work extraction protocols for multitime quantum processes. We

begin with the simplest one, which makes no use of the non-Markovianity of the process and is thus set as a reference for the other ones.

*Sequential optimization*—The first protocol we propose is characterized by sequential optimization of inputs and local extraction of work. As described in Fig. 1(b), for any process  $\mathcal{P} \in \mathbb{P}_n$ , the channel  $\mathcal{P}_1$  associated with its first step is always well defined. Therefore, for this channel, we could apply the protocol from Ref. [13] to extract  $W(\mathcal{P}_1)$  of work in the first step. This is achieved by preparing an optimal state  $\rho_1$  and feeding it to the first step of the process, then extracting work from the first output state  $\mathcal{P}_1(\rho_1)$ . After using  $\rho_1$  as the first input, the second step of the process will be described by a channel  $\mathcal{P}_{2|\rho_1}$ , for which the procedure could be repeated, leading to the extraction of  $W(\mathcal{P}_{2|\rho_1})$  of work. The sequential optimization protocol consists of iterating this procedure until the last step of the process. A rigorous definition for the total work  $W^{\text{seq}}(\mathcal{P})$  extracted through the sequential optimization protocol is given in the Appendix.

*Joint optimization*—While the result of Ref. [13] implies  $W^{\text{seq}}(\mathcal{P})$  is the maximum work extractable from Markovian processes, this does not hold in general in the presence of non-Markovianity. Consider, for example, the process  $\mathcal{P}$  of Fig. 2(a). Notice that the channel  $\mathcal{P}_1$  has the thermal state  $\gamma_S$  as a fixed output, such that no work can be extracted from its output independently of the input. In this case, sequential optimization implies the optimal first input is  $\rho_1 = \gamma_S$ , as it can be prepared for free, yielding  $W(\mathcal{P}_1) = 0$ . This implies the second channel  $\mathcal{P}_{2|\rho_1}$  has  $\mathcal{X}(\gamma_S)$  as a fixed output. Again, the optimal input is  $\rho_2 = \gamma_S$ , yielding  $W^{\text{seq}}(\mathcal{P}) = kTS[\mathcal{X}(\gamma_S)||\gamma_S] = (1 - e^{-E/kT})E$ .

It is possible, however, to make use of the non-Markovianity of the process to extract even more work from it. If the experimenter changes the first input to  $\rho_1 = |0\rangle\langle 0|$ , at a cost of  $kT \ln(1 + e^{-E/kT})$ , the second output will be  $|1\rangle\langle 1|$ , from which one can extract  $E + kT \ln(1 + e^{-E/kT})$  of work, resulting in a net distilled work equal

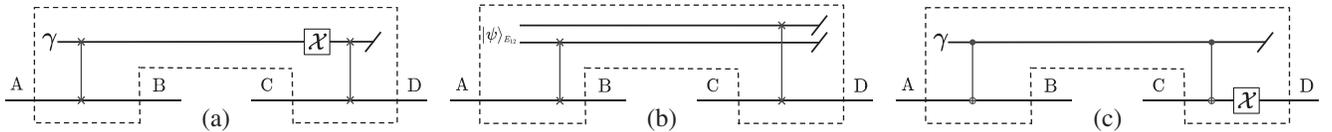


FIG. 2. Examples of non-Markovian processes. (a) A process where sequential optimization is not optimal. Both system and environment are qubits with Hamiltonian  $H = E|1\rangle\langle 1|$ . The initial state of the environment is thermal and the first step of the process is a SWAP gate between system and environment, while the second step is a NOT gate on the environment,  $\mathcal{X}(\bullet) = X \cdot X^\dagger$ ,  $X = |0\rangle\langle 1| + |1\rangle\langle 0|$ , followed by another SWAP gate. (b) A process where no work can be extracted through joint optimization. The environment is composed by two qubits  $E_1$  and  $E_2$  with Hamiltonians  $H_{E_1} = H_{E_2} = E|1\rangle\langle 1|$ . The system is also a qubit with the same Hamiltonian. The initial state of the environment is  $|\psi\rangle_{E_{12}} \propto |00\rangle + e^{-E/2kT}|11\rangle$ , such that it is locally thermal despite being globally pure. Each step of the process is a SWAP interaction between the system and a single part of the environment, resembling a collisional model but allowing for non-Markovianity since the ancillae are initially correlated. (c) A process where global extraction is not optimal. Both system and environment are qubits with Hamiltonian  $H = E|1\rangle\langle 1|$ . The environment is initially thermal at temperature  $T$ . The first step of the dynamics consists of a controlled-NOT (CNOT) gate with control on the environment and target on the system. In the second step we have the same CNOT gate, but this time followed by a NOT gate operation on the system.

to  $E$ . This advantage with respect to the sequential optimization comes from the fact that the non-Markovianity of the process allows the experimenter to spend some work in the first step to extract even more in the second one. This mechanism is what we call *work investment* (WI).

To make full use of work investment, it is necessary to perform a *joint optimization protocol*, in which the inputs of each step of the process are chosen such that the total work distilled is maximized. The work  $W^{\text{joint}}(\mathcal{P})$  extractable through joint optimization is mathematically defined in the Appendix. Since this construction encompasses the cases where sequential optimization is optimal, it is immediately clear that  $W^{\text{joint}}(\mathcal{P})$  is always greater than or equal to  $W^{\text{seq}}(\mathcal{P})$ , with the drawback that in practice the joint optimization is much harder to be computed than the sequential one.

Knowing that sequential optimization is optimal when the process is Markovian, we should expect the advantage of work investment to be little if the amount of non-Markovianity in the process is little. To validate such an intuition, we define the additional work extractable from work investment as  $\Delta W^{\text{WI}}(\mathcal{P}) := W^{\text{joint}}(\mathcal{P}) - W^{\text{seq}}(\mathcal{P})$ , and in Supplemental Material [93] we prove the following continuity bound.

*Theorem 1*—For any  $n$ -step process  $\mathcal{P} \in \mathbb{P}_n$  with non-Markovianity  $N(\mathcal{P})$ , describing the evolution of a system  $S$  with Hamiltonian  $H_S$  and in contact with a bath at temperature  $T$ , it holds that

$$\Delta W^{\text{WI}}(\mathcal{P}) \leq kT \mathcal{F}(H_S, T, n) [N(\mathcal{P})]^{1/4}, \quad (5)$$

where

$$\mathcal{F}(H_S, T, n) = 2^{1/4} [\sqrt{2} \ln 2 + S(\Pi_{E_{\max}} || \gamma_S)] (n-1),$$

and  $\Pi_{E_{\max}} = |E_{\max}\rangle\langle E_{\max}|$  is the projector onto the most energetic eigenstate of  $H_S$ .

Being a continuity bound, the above inequality is not tight in general [94], but it is important to show that the amount of non-Markovianity quantitatively limits the thermodynamic yield obtainable from an initial work investment. This is of practical relevance, since typical processes have small non-Markovianity if the environment is large enough [52].

*Global optimization*—Work investment, however, is not the only mechanism through which non-Markovianity can enhance work extraction. To see this, consider the process of Fig. 2(b). Notice that, independently of the chosen inputs  $\rho_1$  and  $\rho_2$ , the two outputs of the process will locally be thermal states  $\gamma_S$ , from which no work can be extracted. Nevertheless, if the experimenter has access to an auxiliary quantum system, which we call a *side memory* [95], where the first output of the process can be stored until after the second step, it is possible to perform work extraction from the global output state  $\rho_{12} = |\psi\rangle\langle\psi|_{12}$ , obtaining

$W(\rho_{12}) = 2kTS(\gamma) > 0$ . This happens because non-Markovian dynamics may create correlations between outputs at different times, even if the inputs were initially uncorrelated. This mechanism is what we call *multitime correlations* (MTC).

To fully explore the multitime correlations that result from the process, the experimenter must perform a *global optimization protocol*, in which all the outputs are stored until the end and work is distilled from the global final state. The inputs are chosen as to achieve the maximum globally extractable work  $W^{\text{global}}(\mathcal{P})$ . Besides the computational complexity of carrying out such optimizations, the implementation of this protocol is likely to be experimentally challenging in most cases, as the amount of side memory resources scales with  $n$ . Still, from a theoretical standpoint, it shows a second way through which non-Markovianity influences work extraction. Importantly, the work extractable from multitime correlations  $\Delta W^{\text{MTC}}(\mathcal{P}) := W^{\text{global}}(\mathcal{P}) - W^{\text{joint}}(\mathcal{P})$  is directly related to the non-Markovianity of the process through the following result, which is also proven in Supplemental Material [93].

*Theorem 2*—For any  $n$ -step process  $\mathcal{P} \in \mathbb{P}_n$  with non-Markovianity  $N(\mathcal{P})$ , describing the evolution of a system  $S$  in contact with a bath at temperature  $T$ , it holds that

$$\Delta W^{\text{MTC}}(\mathcal{P}) \leq kTN(\mathcal{P}). \quad (6)$$

Another meaningful way to understand the result above is by defining the maximum work  $W_i^{\text{max}}(\mathcal{P})$  locally extractable in the  $i$ -th step of a process  $\mathcal{P}$  as  $W(\mathcal{P}_{i|\mathbf{r}})$  maximized over all possible vectors  $\mathbf{r}$  of previous input states. Theorem 2 then implies that, while for joint optimization we have  $W^{\text{joint}}(\mathcal{P}) \leq \sum_{i=1}^n W_i^{\text{max}}(\mathcal{P})$ , for global extraction we get

$$W^{\text{global}}(\mathcal{P}) \leq \sum_{i=1}^n W_i^{\text{max}}(\mathcal{P}) + kTN(\mathcal{P}). \quad (7)$$

This clearly shows how the global protocol allows for work distillation using not only the local athermalities of the process, but also the multitime correlations arising from its non-Markovianity as extra resource. This is crucial, for example, in the process of Fig. 2(b), in which the  $W_i^{\text{max}}$  are all zero and yet there is nonzero work to be extracted from the correlations. In fact, in Sec. VI of [93] we show that, for the example of Fig. 2(b), the bound given by Theorem 2 is tight.

*Comb optimization*—All three protocols we have discussed so far involve preparing uncorrelated input states and feeding them into each step of the process. However, as shown in Fig. 1(a), we could in principle have general channels connecting consecutive time steps. To see how this generalization may lead to a further advantage in work extraction, we consider the process  $\mathcal{P}$  from Fig. 2(c). By preparing the first input  $\rho_1 = |0\rangle\langle 0|$  and using an identity

channel  $\mathcal{I}$  to connect the two steps, we obtain the final state  $|1\rangle\langle 1|$ , yielding a net extracted work of  $E$ . On the other hand, using a global extraction protocol we strictly have  $W^{\text{global}}(\mathcal{P}) < E$  [93]. The advantage in this case comes from the fact that, in the first step of the dynamics, system and environment create correlations that, despite not being immediately useful for work extraction, allow for the state  $|1\rangle\langle 1|$  to be achievable in the second step, from which a higher amount of work can be distilled. If instead the experimenter stores the first output and prepares a new input, the state  $|1\rangle\langle 1|$  is not achievable anymore because system and environment are not initially correlated. Importantly, this advantage cannot be present in a Markovian dynamics as system-environment correlations are lost in between time steps. For these reasons, the last mechanism through which non-Markovianity powers up work extraction is named *system-environment correlations* (SEC).

The protocol that allows the experimenter to use all system-environment correlations present in a process consists of performing general quantum channels on the system in between the steps of the dynamics. To achieve all possible system transformations, these channels may be connected by an ancilla. Notice that, as shown in Fig. 1(a), this is equivalent to performing a comb  $\mathcal{S}$  on the system, which is the most general mapping from a process tensor to a final state. Therefore, this class of protocols, named *comb optimization*, is the most general possible for extracting work from a multitime quantum process, containing all previously discussed ones.

Note that, for optimizing over combs to obtain the most efficient protocol, we need to use the fact that the work cost of implementing a general nonthermal channel  $\mathcal{E}$  is given by  $W(\mathcal{E})$  [14,15]. With this in mind, we can provide a continuity bound to the work gain  $\Delta W^{\text{SEC}}(\mathcal{P}) := W^{\text{comb}}(\mathcal{P}) - W^{\text{global}}(\mathcal{P})$  obtainable through system-environment correlations [93], where  $W^{\text{comb}}(\mathcal{P})$  is the work extractable from the process  $\mathcal{P}$  using comb optimization, defined in the Appendix.

*Theorem 3*—For any  $n$ -step process  $\mathcal{P} \in \mathbb{P}_n$  with non-Markovianity  $N(\mathcal{P})$ , describing the evolution of a system  $S$  with Hamiltonian  $H_S$  and in contact with a bath at temperature  $T$ , it holds that

$$\Delta W^{\text{SEC}}(\mathcal{P}) \leq kT \mathcal{G}(H_S, T, n) [N(\mathcal{P})]^{1/4}, \quad (8)$$

where

$$\mathcal{G}(H_S, T, n) = 2^{1/4} [\sqrt{2} \ln 2 + (2n - 1) S(\Pi_{E_{\max}} || \gamma_S)],$$

and  $\Pi_{E_{\max}} = |E_{\max}\rangle\langle E_{\max}|$  is the projector onto the most energetic eigenstate of  $H_S$ .

*Conclusion*—We have established the fundamental role of non-Markovianity in thermodynamic work extraction from general quantum processes. We defined a hierarchy of

four classes of extraction protocols [Eq. (1)], which led to the identification of three mechanisms through which non-Markovianity enhances work extraction: work investment, multitime correlations, and system-environment correlations. We presented examples and bounds that show how the advantage these mechanisms provide disappears when the process has a vanishing degree of non-Markovianity. The combined effect of these mechanisms (Theorems 1–3) leads to a remarkable characterization of the work extractable from an  $n$ -step quantum process  $\mathcal{P}$  under the most general comb optimization,

$$W^{\text{comb}}(\mathcal{P}) = W^{\text{seq}}(\mathcal{P}) + \Delta W^N(\mathcal{P}), \quad (9)$$

where  $W^{\text{seq}}$  is the best that one can do in the Markovian case, while  $\Delta W^N := \Delta W^{\text{WI}} + \Delta W^{\text{MTC}} + \Delta W^{\text{SEC}}$  captures the additional work obtainable by non-Markovian resources, which is bound by a *monotonic* function of the non-Markovianity degree  $N(\mathcal{P})$  of the process,  $\Delta W^N(\mathcal{P}) \leq kT [N(\mathcal{P}) + ((3n - 2)S(\Pi_{E_{\max}} || \gamma_S) + \sqrt{2} \ln 2n)(2N(\mathcal{P}))^{1/4}]$ .

Our Letter establishes a new qualitative and quantitative framework to investigate non-Markovian effects in quantum thermodynamics, which may be seen to complement that of Refs. [37–40] in the context of athermality resource theories. Moreover, our results extend those of Ref. [16] and help understand the thermodynamic advantages shown in Refs. [17–32] on a more fundamental level. The bounds we proved here may also be employed to define thermodynamic witnesses of non-Markovianity, and by splitting the temporal correlations into classical and quantum [96–100], one could obtain witnesses of purely quantum memory for general dynamics [62,101]. We expect our results to stimulate further investigation on the interplay between non-Markovianity and thermodynamics, going beyond the asymptotic regime considered here, to get more experimentally feasible estimates of the work cost and yield of implementing quantum processes with finite resources. This may lead to a deeper comprehension of the energetics of quantum systems [102], potentially inspiring optimal designs for sustainable near-term quantum technologies.

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- [1] *Thermodynamics in the Quantum Regime: Fundamental Aspects and New Directions*, edited by F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer International Publishing, Cham, 2018), 10.1007/978-3-319-99046-0.

- [2] M. Perarnau-Llobet, K. V. Hovhannisyan, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, Extractable work from correlations, *Phys. Rev. X* **5**, 041011 (2015).
- [3] K. Korzekwa, M. Lostaglio, J. Oppenheim, and D. Jennings, The extraction of work from quantum coherence, *New J. Phys.* **18**, 023045 (2016).
- [4] G. Francica, J. Goold, F. Plastina, and M. Paternostro, Daemonic ergotropy: Enhanced work extraction from quantum correlations, *npj Quantum Inf.* **3**, 12 (2017).
- [5] B. Morris, L. Lami, and G. Adesso, Assisted work distillation, *Phys. Rev. Lett.* **122**, 130601 (2019).
- [6] B. Yadin, B. Morris, and G. Adesso, Mixing indistinguishable systems leads to a quantum Gibbs paradox, *Nat. Commun.* **12**, 1471 (2021).
- [7] D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth, Thermodynamic cost of reliability and low temperatures: Tightening Landauer’s principle and the second law, *Int. J. Theor. Phys.* **39**, 2717 (2000).
- [8] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource theory of quantum states out of thermal equilibrium, *Phys. Rev. Lett.* **111**, 250404 (2013).
- [9] N. H. Y. Ng and M. P. Woods, Resource theory of quantum thermodynamics: Thermal operations and second laws, in *Thermodynamics in the Quantum Regime: Fundamental Aspects and New Directions*, edited by F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer International Publishing, Cham, 2018), pp. 625–650, 10.1007/978-3-319-99046-0\_26.
- [10] M. Lostaglio, An introductory review of the resource theory approach to thermodynamics, *Rep. Prog. Phys.* **82**, 114001 (2019).
- [11] M. Horodecki and J. Oppenheim, Fundamental limitations for quantum and nanoscale thermodynamics, *Nat. Commun.* **4**, 2059 (2013).
- [12] F. Brandao, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, The second laws of quantum thermodynamics, *Proc. Natl. Acad. Sci. U.S.A.* **112**, 3275 (2015).
- [13] M. Navascués and L. P. García-Pintos, Nonthermal quantum channels as a thermodynamical resource, *Phys. Rev. Lett.* **115**, 010405 (2015).
- [14] P. Faist, M. Berta, and F. Brandão, Thermodynamic capacity of quantum processes, *Phys. Rev. Lett.* **122**, 200601 (2019).
- [15] P. Faist, M. Berta, and F. G. Brandao, Thermodynamic implementations of quantum processes, *Commun. Math. Phys.* **384**, 1709 (2021).
- [16] B. Bylicka, M. Tukiainen, D. Chruściński, J. Piilo, and S. Maniscalco, Thermodynamic power of non-Markovianity, *Sci. Rep.* **6**, 27989 (2016).
- [17] P. Strasberg, G. Schaller, N. Lambert, and T. Brandes, Nonequilibrium thermodynamics in the strong coupling and non-Markovian regime based on a reaction coordinate mapping, *New J. Phys.* **18**, 073007 (2016).
- [18] G. Thomas, N. Siddharth, S. Banerjee, and S. Ghosh, Thermodynamics of non-Markovian reservoirs and heat engines, *Phys. Rev. E* **97**, 062108 (2018).
- [19] M. Pezzutto, M. Paternostro, and Y. Omar, An out-of-equilibrium non-Markovian quantum heat engine, *Quantum Sci. Technol.* **4**, 025002 (2019).
- [20] P. Abiuso and V. Giovannetti, Non-markov enhancement of maximum power for quantum thermal machines, *Phys. Rev. A* **99**, 052106 (2019).
- [21] F. Kamin, F. Tabesh, S. Salimi, F. Kheirandish, and A. C. Santos, Non-Markovian effects on charging and self-discharging process of quantum batteries, *New J. Phys.* **22**, 083007 (2020).
- [22] S. Bhattacharya, B. Bhattacharya, and A. Majumdar, Thermodynamic utility of non-Markovianity from the perspective of resource interconversion, *J. Phys. A* **53**, 335301 (2020).
- [23] M. Debiussac, D. Grass, J. J. Alonso, E. Lutz, and N. Kiesel, Thermodynamics of continuous non-Markovian feedback control, *Nat. Commun.* **11**, 1360 (2020).
- [24] O. Abah and M. Paternostro, Implications of non-Markovian dynamics on information-driven engine, *J. Phys. Commun.* **4**, 085016 (2020).
- [25] P. A. Camati, J. F. G. Santos, and R. M. Serra, Employing non-Markovian effects to improve the performance of a quantum Otto refrigerator, *Phys. Rev. A* **102**, 012217 (2020).
- [26] Y. Shirai, K. Hashimoto, R. Tezuka, C. Uchiyama, and N. Hatano, Non-Markovian effect on quantum Otto engine: Role of system-reservoir interaction, *Phys. Rev. Res.* **3**, 023078 (2021).
- [27] N. Yunger Halpern and D. T. Limmer, Fundamental limitations on photoisomerization from thermodynamic resource theories, *Phys. Rev. A* **101**, 042116 (2020).
- [28] G. Spaventa, S. F. Huelga, and M. B. Plenio, Capacity of non-Markovianity to boost the efficiency of molecular switches, *Phys. Rev. A* **105**, 012420 (2022).
- [29] K. Ptaszyński, Non-Markovian thermal operations boosting the performance of quantum heat engines, *Phys. Rev. E* **106**, 014114 (2022).
- [30] J. W. Cheong, A. Pradana, and L. Y. Chew, Effects of non-Markovianity on daemonic ergotropy in the quantum switch, *Phys. Rev. A* **108**, 012201 (2023).
- [31] I. A. Picatoste, A. Colla, and H.-P. Breuer, Dynamically emergent quantum thermodynamics: Non-Markovian Otto cycle, *Phys. Rev. Res.* **6**, 013258 (2024).
- [32] A. Mohammadi and A. Shafiee, Quantum non-Markovianity, quantum coherence and extractable work in a general quantum process, *Phys. Chem. Chem. Phys.* **26**, 3990 (2024).
- [33] B. Vacchini, Non-Markovian master equations from piecewise dynamics, *Phys. Rev. A* **87**, 030101(R) (2013).
- [34] A. Das, A. Bera, S. Chakraborty, and D. Chruściński, Thermodynamics and the quantum speed limit in the non-Markovian regime, *Phys. Rev. A* **104**, 042202 (2021).
- [35] A. Colla and H.-P. Breuer, Open-system approach to nonequilibrium quantum thermodynamics at arbitrary coupling, *Phys. Rev. A* **105**, 052216 (2022).
- [36] A. Rivas, Strong coupling thermodynamics of open quantum systems, *Phys. Rev. Lett.* **124**, 160601 (2020).
- [37] M. Lostaglio, Á. M. Alhambra, and C. Perry, Elementary thermal operations, *Quantum* **2**, 52 (2018).
- [38] M. Lostaglio and K. Korzekwa, Continuous thermomajorization and a complete set of laws for Markovian thermal processes, *Phys. Rev. A* **106**, 012426 (2022).

- [39] K. Korzekwa and M. Lostaglio, Optimizing thermalization, *Phys. Rev. Lett.* **129**, 040602 (2022).
- [40] J. Czartowski, A. de Oliveira Junior, and K. Korzekwa, Thermal recall: Memory-assisted Markovian thermal processes, *PRX Quantum* **4**, 040304 (2023).
- [41] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, Non-Markovian quantum processes: Complete framework and efficient characterization, *Phys. Rev. A* **97**, 012127 (2018).
- [42] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Theoretical framework for quantum networks, *Phys. Rev. A* **80**, 022339 (2009).
- [43] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, Operational Markov condition for quantum processes, *Phys. Rev. Lett.* **120**, 040405 (2018).
- [44] Á. Rivas, S. F. Huelga, and M. B. Plenio, Quantum non-Markovianity: Characterization, quantification and detection, *Rep. Prog. Phys.* **77**, 094001 (2014).
- [45] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Colloquium: Non-Markovian dynamics in open quantum systems, *Rev. Mod. Phys.* **88**, 021002 (2016).
- [46] I. de Vega and D. Alonso, Dynamics of non-Markovian open quantum systems, *Rev. Mod. Phys.* **89**, 015001 (2017).
- [47] L. Li, M. J. Hall, and H. M. Wiseman, Concepts of quantum non-Markovianity: A hierarchy, *Phys. Rep.* **759**, 1 (2018).
- [48] S. Milz, M. Kim, F. A. Pollock, and K. Modi, Completely positive divisibility does not mean Markovianity, *Phys. Rev. Lett.* **123**, 040401 (2019).
- [49] D. Chruściński, Dynamical maps beyond Markovian regime, *Phys. Rep.* **992**, 1 (2022).
- [50] P. Taranto, F. A. Pollock, S. Milz, M. Tomamichel, and K. Modi, Quantum Markov order, *Phys. Rev. Lett.* **122**, 140401 (2019).
- [51] P. Taranto, S. Milz, F. A. Pollock, and K. Modi, Structure of quantum stochastic processes with finite Markov order, *Phys. Rev. A* **99**, 042108 (2019).
- [52] P. Figueroa-Romero, K. Modi, and F. A. Pollock, Almost Markovian processes from closed dynamics, *Quantum* **3**, 136 (2019).
- [53] S. Milz, F. Sakuldee, F. A. Pollock, and K. Modi, Kolmogorov extension theorem for (quantum) causal modelling and general probabilistic theories, *Quantum* **4**, 255 (2020).
- [54] S. Milz, D. Egloff, P. Taranto, T. Theurer, M. B. Plenio, A. Smirne, and S. F. Huelga, When is a non-Markovian quantum process classical?, *Phys. Rev. X* **10**, 041049 (2020).
- [55] S. Milz, C. Spee, Z.-P. Xu, F. A. Pollock, K. Modi, and O. Gühne, Genuine multipartite entanglement in time, *SciPost Phys.* **10**, 141 (2021).
- [56] P. Taranto, F. A. Pollock, and K. Modi, Non-Markovian memory strength bounds quantum process recoverability, *npj Quantum Inf.* **7**, 149 (2021).
- [57] P. Figueroa-Romero, F. A. Pollock, and K. Modi, Markovianization with approximate unitary designs, *Commun. Phys.* **4**, 127 (2021).
- [58] F. Sakuldee, P. Taranto, and S. Milz, Connecting commutativity and classicality for multitime quantum processes, *Phys. Rev. A* **106**, 022416 (2022).
- [59] M. Capela, L. C. Céleri, R. Chaves, and K. Modi, Quantum Markov monogamy inequalities, *Phys. Rev. A* **106**, 022218 (2022).
- [60] P. Taranto, T. J. Elliott, and S. Milz, Hidden quantum memory: Is memory there when somebody looks?, *Quantum* **7**, 991 (2023).
- [61] G. D. Berk, A. J. P. Garner, B. Yadin, K. Modi, and F. A. Pollock, Resource theories of multi-time processes: A window into quantum non-Markovianity, *Quantum* **5**, 435 (2021).
- [62] P. Taranto, M. T. Quintino, M. Murao, and S. Milz, Characterising the hierarchy of multi-time quantum processes with classical memory, *Quantum* **8**, 1328 (2024).
- [63] G. Zambon and D. O. Soares-Pinto, Relations between Markovian and non-Markovian correlations in multitime quantum processes, *Phys. Rev. A* **109**, 062401 (2024).
- [64] L. Santos, Z.-P. Xu, J. Piilo, and O. Gühne, Quantifying information flow in quantum processes, [arXiv:2402.04213](https://arxiv.org/abs/2402.04213).
- [65] S. Milz, F. A. Pollock, and K. Modi, Reconstructing non-Markovian quantum dynamics with limited control, *Phys. Rev. A* **98**, 012108 (2018).
- [66] G. A. White, C. D. Hill, F. A. Pollock, L. C. Hollenberg, and K. Modi, Demonstration of non-Markovian process characterisation and control on a quantum processor, *Nat. Commun.* **11**, 6301 (2020).
- [67] G. A. White, F. A. Pollock, L. C. Hollenberg, K. Modi, and C. D. Hill, Non-Markovian quantum process tomography, *PRX Quantum* **3**, 020344 (2022).
- [68] G. White, Characterization and control of non-Markovian quantum noise, *Nat. Rev. Phys.* **4**, 287 (2022).
- [69] G. A. White, K. Modi, and C. D. Hill, Filtering crosstalk from bath non-Markovianity via spacetime classical shadows, *Phys. Rev. Lett.* **130**, 160401 (2023).
- [70] I. A. Aloisio, G. A. White, C. D. Hill, and K. Modi, Sampling complexity of open quantum systems, *PRX Quantum* **4**, 020310 (2023).
- [71] M. R. Jørgensen and F. A. Pollock, Exploiting the causal tensor network structure of quantum processes to efficiently simulate non-Markovian path integrals, *Phys. Rev. Lett.* **123**, 240602 (2019).
- [72] M. R. Jørgensen and F. A. Pollock, Discrete memory kernel for multitime correlations in non-Markovian quantum processes, *Phys. Rev. A* **102**, 052206 (2020).
- [73] M. Cygorek, M. Cosacchi, A. Vagov, V. M. Axt, B. W. Lovett, J. Keeling, and E. M. Gauger, Simulation of open quantum systems by automated compression of arbitrary environments, *Nat. Phys.* **18**, 662 (2022).
- [74] P. Fowler-Wright, B. W. Lovett, and J. Keeling, Efficient many-body non-Markovian dynamics of organic polaritons, *Phys. Rev. Lett.* **129**, 173001 (2022).
- [75] D. Gribben, A. Strathearn, G. E. Fux, P. Kirton, and B. W. Lovett, Using the environment to understand non-Markovian open quantum systems, *Quantum* **6**, 847 (2022).
- [76] M. Cygorek, J. Keeling, B. W. Lovett, and E. M. Gauger, Sublinear scaling in non-Markovian open quantum systems simulations, *Phys. Rev. X* **14**, 011010 (2024).
- [77] G. E. Fux, D. Kilda, B. W. Lovett, and J. Keeling, Tensor network simulation of chains of non-Markovian open quantum systems, *Phys. Rev. Res.* **5**, 033078 (2023).

- [78] P. Strasberg, Repeated interactions and quantum stochastic thermodynamics at strong coupling, *Phys. Rev. Lett.* **123**, 180604 (2019).
- [79] P. Figueroa-Romero, K. Modi, and F. A. Pollock, Equilibration on average in quantum processes with finite temporal resolution, *Phys. Rev. E* **102**, 032144 (2020).
- [80] Z. Huang, Fluctuation theorems for multitime processes, *Phys. Rev. A* **105**, 062217 (2022).
- [81] Z. Huang, Multiple entropy production for multitime quantum processes, *Phys. Rev. A* **108**, 032217 (2023).
- [82] N. Dowling, P. Figueroa-Romero, F. A. Pollock, P. Strasberg, and K. Modi, Relaxation of multitime statistics in quantum systems, *Quantum* **7**, 1027 (2023).
- [83] N. Dowling, P. Figueroa-Romero, F. A. Pollock, P. Strasberg, and K. Modi, Equilibration of multitime quantum processes in finite time intervals, *SciPost Phys. Core* **6**, 043 (2023).
- [84] P. Figueroa-Romero, K. Modi, R. J. Harris, T. M. Stace, and M.-H. Hsieh, Randomized benchmarking for non-Markovian noise, *PRX Quantum* **2**, 040351 (2021).
- [85] P. Figueroa-Romero, K. Modi, and M.-H. Hsieh, Towards a general framework of randomized benchmarking incorporating non-Markovian noise, *Quantum* **6**, 868 (2022).
- [86] G. D. Berk, S. Milz, F. A. Pollock, and K. Modi, Extracting quantum dynamical resources: Consumption of non-Markovianity for noise reduction, *npj Quantum Inf.* **9**, 104 (2023).
- [87] P. Figueroa-Romero, M. Papič, A. Auer, M.-H. Hsieh, K. Modi, and I. de Vega, Operational Markovianization in randomized benchmarking, *Quantum Sci. Technol.* **9**, 035020 (2024).
- [88] E. P. Butler, G. E. Fux, C. Ortega-Taberner, B. W. Lovett, J. Keeling, and P. R. Eastham, Optimizing performance of quantum operations with non-Markovian decoherence: The tortoise or the hare?, *Phys. Rev. Lett.* **132**, 060401 (2024).
- [89] G. Zambon, Process tensor distinguishability measures, *Phys. Rev. A* **110**, 042210 (2024).
- [90] A naive approach here would be to try and compare the distillable works of a given process and its Markovian version. The problem is that for general process tensors this is not uniquely defined. A common approach [43,61–63,86] is to use the process whose Choi state is closer in relative entropy. However, the closest Markov Choi state is different for different measures, such as the trace distance [52] or other Schatten norms [84]. Moreover, distinguishing process tensors by their Choi states may be problematic from a resource-theoretical perspective [89].
- [91] If the Hamiltonian is explicitly time dependent, we need to account for the work cost of the drive: any change in  $H_S(t)$  can be decomposed into a sequence of energy-level shifts (which require work input) interlaced with thermalizations. Our results show that non-Markovianity can still enhance the gross work yield even with a driven Hamiltonian, but any energy expended to modulate the Hamiltonian must be subtracted from that yield. The thermodynamic bounds we derive (ultimately given by free-energy differences) would remain appropriate upper limits on *net* work extraction.
- [92] D. Kretschmann and R. F. Werner, Quantum channels with memory, *Phys. Rev. A* **72**, 062323 (2005).
- [93] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.134.200401> for additional proofs and derivations.
- [94] A. Bluhm, Á. Capel, P. Gondolf, and A. Pérez-Hernández, Continuity of quantum entropic quantities via almost convexity, *IEEE Trans. Inf. Theory* **69**, 5869 (2023).
- [95] This auxiliary system is considered ideal for internal consistency of the framework. Note that the system of interest is anything the experimenter has access to that also interacts with an uncontrolled environment, represented by the black horizontal line on the top of Fig. 1(a). On the other hand, the ancilla is, by definition, only controlled by the experimenter, as represented by the black horizontal line on the bottom of Fig. 1(a).
- [96] W. H. Zurek, Quantum discord and Maxwell’s demons, *Phys. Rev. A* **67**, 012320 (2003).
- [97] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Thermodynamical approach to quantifying quantum correlations, *Phys. Rev. Lett.* **89**, 180402 (2002).
- [98] P. Liuzzo-Scorpo, L. A. Correa, R. Schmidt, and G. Adesso, Thermodynamics of quantum feedback cooling, *Entropy* **18**, 48 (2016).
- [99] G. Adesso, T. R. Bromley, and M. Cianciaruso, Measures and applications of quantum correlations, *J. Phys. A* **49**, 473001 (2016).
- [100] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, Cambridge, England, 2013).
- [101] C. Giarmatzi and F. Costa, Witnessing quantum memory in non-Markovian processes, *Quantum* **5**, 440 (2021).
- [102] A. Auffèves, Quantum technologies need a quantum energy initiative, *PRX Quantum* **3**, 020101 (2022).

## End Matter

*Appendix: Definitions*—We denote by  $\mathcal{P}_{i|r}$  the quantum channel describing the  $i$ -th step of the process  $\mathcal{P} \in \mathbb{P}_n$ , given the inputs of previous steps were the first  $i-1$  entries of the vector  $\mathbf{r} = (\rho_1, \dots, \rho_{i-1}, \dots, \rho_n)$ . Then, let  $\rho_{i|r}^{\max}$  be the state maximizing the work extractable from the channel  $\mathcal{P}_{i|r}$ , that is,

$$\rho_{i|r}^{\max} := \operatorname{argmax}_{\rho} [W(\mathcal{P}_{i|r}(\rho)) - W(\rho)]. \quad (\text{A1})$$

This allows for the following definition.

*Definition A1*—Let  $\mathbf{r}^{\text{seq}}$  be the vector of states sequentially optimizing the work extraction from a process  $\mathcal{P} \in \mathbb{P}_n$ , that is,  $\mathbf{r}_1^{\text{seq}} = \rho_1^{\max}$  and  $\mathbf{r}_i^{\text{seq}} = \rho_{i|(\rho_1^{\text{seq}}, \dots, \rho_{i-1}^{\text{seq}})}^{\max}$ .

The total work  $W^{\text{seq}}(\mathcal{P})$  extractable from a process  $\mathcal{P} \in \mathbb{P}_n$  through the sequential optimization protocol is given by

$$W^{\text{seq}}(\mathcal{P}) := \sum_{i=1}^n W(\mathcal{P}_{i|r^{\text{seq}}}). \quad (\text{A2})$$

The joint optimization protocol is given by relaxing the sequential optimization to a joint one, as follows.

*Definition A2*—The total work  $W^{\text{joint}}(\mathcal{P})$  extractable from a process  $\mathcal{P} \in \mathbb{P}_n$  through the joint optimization protocol is given by

$$W^{\text{joint}}(\mathcal{P}) := \max_{\mathbf{r}} \sum_{i=1}^n W(\mathcal{P}_{i|r}), \quad (\text{A3})$$

where the maximization is over all possible vectors  $\mathbf{r}$  of inputs.

For the global optimization, one can define a class of combs that implement it. Let  $\mathbb{S}_n^{\text{global}} \subset \mathbb{S}_n$  be the set of combs consisting of inputting state  $\mathbf{r}_i = \rho_i$  to the  $i$ -th step and storing all the outputs until the end. Any such comb  $\mathcal{S}_r$  is uniquely defined by the vector  $\mathbf{r}$  of states it inputs to the process. The implementation cost  $W(\mathcal{S}_r)$  is simply  $\sum_{i=1}^n W(\rho_i)$  and the work extractable in the end is  $W[\mathcal{P}(\mathcal{S}_r)]$ , leading to the following definition.

*Definition A3*—The total work  $W^{\text{global}}(\mathcal{P})$  extractable from a process  $\mathcal{P} \in \mathbb{P}_n$  through the global optimization protocol is given by

$$W^{\text{global}}(\mathcal{P}) := \max_{\mathcal{S}_r \in \mathbb{S}_n^{\text{global}}} \{W[\mathcal{P}(\mathcal{S}_r)] - W(\mathcal{S}_r)\}. \quad (\text{A4})$$

Finally, to define comb optimization we just need to clarify the cost of implementing general combs. Unlike the case of states and channels, it is not known whether the work cost of implementing a comb is asymptotically equal to the work extractable from it. Since any comb  $\mathcal{S} \in \mathbb{S}_n$  can be dilated as the action of  $n$  channels  $\mathcal{E}_i$  acting both on the system and an ancilla initially in the state  $\sigma_A$ , we take the cost  $W(\mathcal{S})$  to be simply  $W(\sigma_A) + \sum_{i=1}^n W(\mathcal{E}_i)$  minimized over all possible dilations of  $\mathcal{S}$ . This leads us to the last definition.

*Definition A4*—The total work  $W^{\text{comb}}(\mathcal{P})$  extractable from a process  $\mathcal{P} \in \mathbb{P}_n$  through comb optimization is given by

$$W^{\text{comb}}(\mathcal{P}) := \max_{\mathcal{S} \in \mathbb{S}_n} \{W[\mathcal{P}(\mathcal{S})] - W(\mathcal{S})\}. \quad (\text{A5})$$